LECTURE 19

ZIQUAN YANG

Proof of Stokes' Theorem for a Parametrized Surface

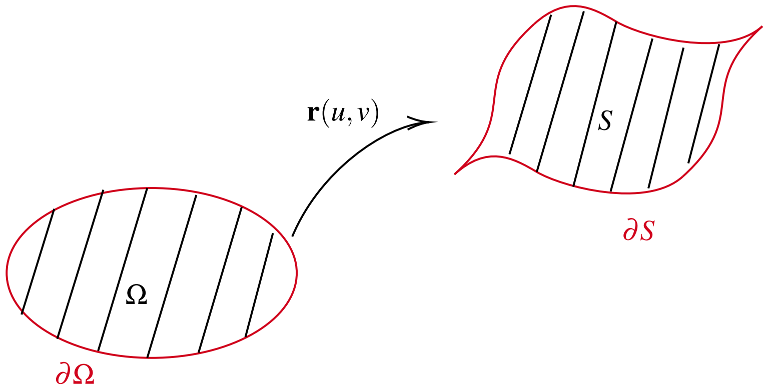
Recall the statement of Stokes' theorem:

Theorem 1. *Let S be a smooth oriented surface having a (piecewise) smooth boundary curve C. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then*

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$$

where $\mathbf{T}$ is unit tangent vector field on the boundary ∂S in the counterclockwise orientation with respect to $\mathbf{n}$.

Let us prove this in a relatively simple senario. Then you see that it follows from Green's theorem, which boils down to the fundamental theorem of calculus. Assume that S is parametrized by $\mathbf{r} : \Omega \to \mathbb{R}^3$, where Ω is a region in $\mathbb{R}^2$ with boundary $\partial\Omega$, and ∂S is nothing but the image of $\partial\Omega$ under $\mathbf{r}$.



Let $t \mapsto \mathbb{R}^2$ for $t \in [a,b]$ be a parametrization for $\partial\Omega$. Then $\partial\Omega$ is parametrized by $t \mapsto \mathbf{r}(u(t), v(t))$, and we have

$$\oint_{\partial S} \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}(\mathbf{r}(u(t), v(t))) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \frac{du}{dt} + \frac{\partial \mathbf{r}}{\partial v} \frac{dv}{dt} \right) dt.$$

Write $\mathbf{G}(u,v)$ for $\mathbf{F}(\mathbf{r}(u,v))$, we can view the right hand side as the integral of an 1-form on $\partial\Omega$:

$$\oint_{\partial\Omega} \left(\mathbf{G} \cdot \frac{\partial \mathbf{r}}{\partial u} \right) du + \left(\mathbf{G} \cdot \frac{\partial \mathbf{r}}{\partial v} \right) dv.$$

Note that the integrands are dot products of two vectors in $\mathbb{R}^3$, and are functions in u, v.

By Green's theorem in Ω, this equals:

$$\iint_\Omega \left[\frac{\partial}{\partial u} \left(\mathbf{G} \cdot \frac{\partial \mathbf{r}}{\partial v} \right) - \frac{\partial}{\partial v} \left(\mathbf{G} \cdot \frac{\partial \mathbf{r}}{\partial u} \right) \right] du dv. \tag{1}$$

Compute the partial derivatives using the product rule:

$$\frac{\partial}{\partial u} \left(\mathbf{G} \cdot \frac{\partial \mathbf{r}}{\partial v} \right) = \frac{\partial \mathbf{G}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} + \mathbf{G} \cdot \frac{\partial^2 \mathbf{r}}{\partial u \partial v},$$

$$\frac{\partial}{\partial v} \left(\mathbf{G} \cdot \frac{\partial \mathbf{r}}{\partial u} \right) = \frac{\partial \mathbf{G}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial u} + \mathbf{G} \cdot \frac{\partial^2 \mathbf{r}}{\partial v \partial u}.$$

Date: April 3, 2025.

Noting (as the surface is smooth[1])

$$\frac{\partial^2 \mathbf{r}}{\partial u \partial v} = \frac{\partial^2 \mathbf{r}}{\partial v \partial u},$$

the integrand in (1) simplifies to

$$\frac{\partial \mathbf{G}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} - \frac{\partial \mathbf{G}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial u}.$$

The derivatives of $\mathbf{G}$ are:

$$\frac{\partial \mathbf{G}}{\partial u} = (\nabla \mathbf{F}) \frac{\partial \mathbf{r}}{\partial u}, \quad \frac{\partial \mathbf{G}}{\partial v} = (\nabla \mathbf{F}) \frac{\partial \mathbf{r}}{\partial v},$$

where $\nabla \mathbf{F}$ is the Jacobian matrix of $\mathbf{F}$, namely,

$$\nabla \mathbf{F} = \begin{bmatrix} \nabla M \\ \nabla N \\ \nabla P \end{bmatrix} = \begin{bmatrix} \partial M/\partial x & \partial M/\partial y & \partial M/\partial z \\ \partial N/\partial x & \partial N/\partial y & \partial N/\partial z \\ \partial P/\partial x & \partial P/\partial y & \partial P/\partial z \end{bmatrix}$$

The integrand becomes:

$$\left((\nabla \mathbf{F}) \frac{\partial \mathbf{r}}{\partial u} \right) \cdot \frac{\partial \mathbf{r}}{\partial v} - \left((\nabla \mathbf{F}) \frac{\partial \mathbf{r}}{\partial v} \right) \cdot \frac{\partial \mathbf{r}}{\partial u} = (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right).$$

You can verify this identity by brute force. After you expand and simplify both sides of the equation, you should get on both sides.

$$(P_y - N_z)(y_u z_v - z_u y_v) + (M_z - P_x)(z_u x_v - x_u z_v) + (N_x - M_y)(x_u y_v - y_u x_v)$$

The normal vector and surface element are:

$$\mathbf{n}\, d\sigma = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\, du\, dv.$$

Thus, the integral becomes:

$$\iint_\Omega (\nabla \times \mathbf{F}(\mathbf{r}(u,v))) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du\, dv = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$

Voila! We are done.

I do not expect that you memorize this proof. However, it is helpful to go through this proof once.

[1]In class I said this is because the boundary is smooth by a slip of tongue.